Properties of Toeplitz Graphs

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A Toeplitz matrix or diagonal-constant matrix, named after Otto Toeplitz, is a matrix in which each descending diagonal from left to right is constant. For instance, the following matrix is a Toeplitz matrix:

\[
\begin{pmatrix}
  a & b & c & d & e \\
  f & a & b & c & d \\
  g & f & a & b & c \\
  h & g & f & a & b \\
  i & h & g & f & a \\
\end{pmatrix}
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A Toeplitz graph $T_n\langle t_1, t_2, \ldots, t_k \rangle$ is a (undirected) graph whose vertex set is $\{1, 2, \ldots, n\}$ and, there is an edge between the vertices $i$ and $j$ iff $|j - i| = t_l$ for some $l = 1, 2, \ldots, k$.

It is easily seen that a Toeplitz graph has a $(0, 1)$ symmetric Toeplitz adjacency matrix.
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Any symmetric Toeplitz Matrix can be determined by it’s first row. If the first row of a $(0, 1)$ symmetric Toeplitz matrix is $\delta$ then we denote the corresponding Toeplitz graph with $T^\delta$, or $T_n\langle t_1, t_2, \ldots, t_k \rangle$, where $n$ is length of $\delta$, and $t_i + 1$’s are the place of non-zero arrays of $\delta$. 
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Toeplitz matrix
Examples

$T_8\langle 3, 5 \rangle$

$T_{22}\langle 3, 4 \rangle$
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Connectivity [C. Heuberger]

- The graph $T_{cn} \langle ct_1, ct_2, \ldots, ct_k \rangle$ is the $c$ disjoint copies of graphs isomorphic to $T_n \langle t_1, t_2, \ldots, t_k \rangle$.

- Let $T_n \langle a, b \rangle$ be a Toeplitz graph with $1 \leq a < b < n$,
  - If $\gcd(a, b) > 1$, or $a + b \geq n + 2$, then $T_n \langle a, b \rangle$ is not connected.
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  - If \( \gcd(a, b) = 1 \), \( a + b \leq n + 1 \), then \( T_n \langle a, b \rangle \) is connected.
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Let $T_n\langle a_1, a_2, \ldots, a_m \rangle$ be a Toeplitz graph and $d_k := \gcd(a_1, \ldots, a_k)$ for $k = 1, \ldots, m$. If $d_m = 1$, and $d_k + a_{k+1} \leq n + 1$, for $k = 1, \ldots, m - 1$, then $T_n\langle a_1, a_2, \ldots, a_m \rangle$ is connected.
Let $n, a, b$ are all odd. Then $T_{n\langle a, b \rangle}$ is non-Hamiltonian.

Suppose $n, a, b$, are not all odd, with $b \equiv 1(\text{mod}2a)$, $n \geq 5b$, if $n$ is even, and $n \geq 6b + a$, if $n$ is odd. Then $T_{n\langle a, b \rangle}$ is Hamiltonian.
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Hamiltonian Toeplitz graphs [C. Heuberger]

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Bipartite Toeplitz Graphs

For $\alpha \in \mathbb{N}$ let $B^\alpha$ denote the infinite 0 – 1 sequence,

$$ (0 \ldots 0 \underbrace{10 \ldots 10}_{\alpha} \ldots 0 \underbrace{10 \ldots 10}_{2\alpha} \ldots 0 \ldots 0 \ldots \ldots) . $$

Theorem [Euler]. An infinite sequence $I$ induce a bipartite Toeplitz graph iff $I$ is dominated by one of the sequences $B^\alpha$, where $\alpha \in \{1, 2, 4, 8, \ldots\}$. 
Bipartite Toeplitz Graphs

In the graph $T_n(a, b)$, let $\alpha = a - 1$, and $\delta = b - a$.

- If $n \leq 2\alpha$, then $S$ is dominated by the sequence $(0 \ldots 0 1 \ldots 1)^{\alpha}$, which is easily shown to induce a bipartite Toeplitz graph.

- If $\alpha = (2\beta + 1)2^r$ with $\beta \in \mathbb{N}$ and if $2^{r+1}$ divides $\delta$, then $S$ is again dominated by a finite subsequence of $B^{\alpha}$, $\alpha = 2^r$, and therefore induces a bipartite Toeplitz graph, too.
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- If $\alpha = (2\beta + 1)2^r$ with $\beta \in \mathbb{N}$ and if $2^{r+1}$ divides $\delta$, then $S$ is again dominated by a finite subsequence of $B^\alpha$, $\alpha = 2^r$, and therefore induces a bipartite Toeplitz graph, too.
If however, $2^{r+1}$ does not divide $\delta$, and if $n > 2\alpha + \delta - \gcd(\alpha, \delta)$, then $T_n$ is not bipartite.

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Planarity, [Euler]

- Infinite case: An infinite \((0 - 1)\)-sequence \(S\) defines a planar Toeplitz graph if and only if \(S\) is dominated by a \((0 - 1)\)-sequence whose 1-entries are at positions \(1 + t_1\), \(1 + t_2\) and \(1 + (t_1 + t_2)\).

- Consequently, for infinite, planar Toeplitz graphs \(T_\infty \langle t_1, t_2, \ldots, t_k \rangle\), \(k\) can’t be more than 3.
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- Finite case: $k$ can be arbitrarily large,
- If $T_n\langle t_1, t_2, \ldots, t_k \rangle$ is planar, and $c \in \mathbb{N}$, then $T_{cn}\langle ct_1, ct_2, \ldots, ct_k, cn - 1 \rangle$ is planar.
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A Greedy type algorithm

1. set $I = \emptyset$, $V^* = V$, goto 2.;
2. choose $i^* = \min\{i : i \in V^*\}$, set $I := I \cup i^*$, goto 3.;
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- $T_n\langle a \rangle$
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The algorithm gives a maximum independent set in $T = T_n\langle s, t \rangle$, for all $n$, if and only if $s = 1$, or $t = ks \pm 1$, where $k$ is an odd integer.
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MAT:
(i): $n = |V(T)|$, $q =$number of layers of $T_n \langle s, t \rangle$, $M = \emptyset$.
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(iv): $T = T_n\langle s, t \rangle \setminus M$, which is again a Toeplitz graph, GO TO (i).
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